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On pseudo-convergent sequences

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by

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A sequence $\{a_i\}$ of elements of a non-Archimedean valuated field K is called pseudo-convergent if either $|a_{i+1} - a_i| < |a_i - a_{i-1}|$ for all $i \geq i_0$ or $a_i = a_{i+1}$ for all $i \geq i_1$. It follows that for such a sequence either i: $|a_i| = |a_{i+1}|$ for all $i \geq i_2$ or ii: $|a_i| > |a_{i+1}|$ for all $i \geq i_3$. The sequence is called (pseudo-convergent) of the first kind or second kind according as i or ii holds. The property ii is sufficient for the pseudo-convergency of a sequence, property i however is not. This concept is due to A. Ostrowski (Untersuchungen zur arithmetischen Theorie der Körper; Math. Zeitschr. 39, p. 269-404 (1934)), who proved the following theorem: If $\{a_i\}$ is a pseudo-convergent sequence ($a_i \in K$) and $f(x)$ a polynomial with coefficients in K then the sequence $\{f(a_i)\}$ is also pseudo-convergent.

F. Loonstra (Pseudokonvergente Folgen in nichtarchimedisch bewerteten Körpern; Proc. Ned. Akad. v. Wet. XLV, p. 913-917 (1942)) treated this problem without using algebraic extension of K , contrary to Ostrowski.

The lemma stated as "Satz IV" however is false, as is shown by the following counter example. Take for K the field of the rational numbers with the 2-adic valuation and let $\bar{K}$ be its completion i.e. the field of 2-adic numbers.

The polynomial $x^2 + 7$ has a zero in $\bar{K}$ and not in K (for the underlying theory see B.L. van der Waerden, Moderne Algebra I, 3e Aufl. (1950) § 79).

Let $a = \sum_{j=0}^{\infty} 2^{\nu_j}$ (with $\nu_{j+1} > \nu_j$) be the 2-adic expansion of such a zero. Now put $a_i = \sum_{j=0}^i 2^{\nu_j}$, then $\{a_i\}$ is pseudo-convergent since $|a_{i+1} - a_i| = 2^{-\nu_{i+1}} < 2^{-\nu_i} = |a_i - a_{i-1}|$.

.) Satz IV. Sei $\{a_i\}$ eine pseudokonvergente Folge. Es gebe weiter in K kein Element α derart, dass die pseudokonvergente Folge $\{a_i - \alpha\}$ von der 2. Art ist. Sei $f(x)$ ein Polynom mit Koeffizienten aus K . Dann ist $|f(a_i)|$ konstant von einem gewissen i_0 an.

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$\{a_i - \alpha\}$ is pseudo-convergent of the first kind for all $\alpha \in K$. For, if $\alpha = \sum_{j=0}^{\infty} 2^{\mu_j}$ (with $\mu_{j+1} > \mu_j$) and if j_0 is the smallest index such that $\mu_{j_0} \neq \nu_{j_0}$ (it always exists because $\alpha \notin K$), then for $i > j_0$ we have

$$|a_i - \alpha| = 2^{-\min(\mu_{j_0}, \nu_{j_0})}.$$

Take for $f(x)$ the polynomial $x^2 + 7$. The sequence $\{a_i^2 + 7\}$ is convergent with the limit 0, hence $|a_i^2 + 7| \rightarrow 0$ and since $|a_i^2 + 7| \neq 0$ we have a contradiction with "Satz IV".